Generalized Hankel Interaction Index Array for Control Structure Selection for Discrete-Time MIMO Bilinear Processes and Plants

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Abstract—The control technology has been orientated towards decentralized and partially decentralized control strategies. To ensure the success of a decentralized or a partially decentralized control in practice, the first necessary step is to determine a suitable control structure. The control structure selection which is the task of selecting suitable input and output pairs for control design is therefore very important. This paper focuses on control structure selection for discrete-time MIMO bilinear processes and plants. The existence of the generalized cross-gramian is studied and it is shown that if the cross-gramian exists, it can be obtained by solving a generalized Sylvester equation. To solve the generalized Sylvester equation in a computationally efficient way, an iterative method is developed and presented. The generalized cross-gramians are computed for all SISO subsystems of the discrete-time MIMO bilinear systems. These gramians are used to build the generalized Hankel Interaction Index Array (HIIA) which is used for control structure selection.

The proposed method for control structure selection is among the few methods supporting bilinear processes and plants, enjoys the advantages of gramian based methods and it is more efficient in terms of computations compared to its counterparts.

I. INTRODUCTION

The control technology has been orientated towards decentralized and partially decentralized control strategies. The over-increasing complexity of industrial processes and plants, limitations and issues related to computations, communications and reliability limit the centralized control strategies in practice, and prevent them to perform effectively and to maintain economic for a lot of today’s processes and plants. The decentralized controllers however, are more convenient for operators to work with [1], [2]. To ensure the success of a decentralized or a partially decentralized control in practice, the first necessary step is to determine a suitable control structure. The control structure selection which is the task of selecting suitable input and output pairs for control design is therefore very important.

Two families of methods have been proposed so far for control structure selection: RGA-based family and Gramian-based family. RGA is a well-known interaction measure which was first introduced in [3]. RGA contains information of dynamics at a particular frequency and is delay insensitive. However, over the years it have been subject of the extensive research [2].

A method from the second category of control structure selection was first proposed in [4] and later in [5]. In this family of methods, the observability and the controllability gramians of each elementary subsystems are computed and used to build a matrix which called the Participation Matrix (PM). A suitable controller structure can be determined by studying elements of PM. The Hankel Interaction Index Array (HIIA) is a similar interaction measure, which was introduced in [6] and it was extended later to support uncertain systems in [7]. The gramian-based control structure selection techniques have several advantages over the the methods from the first category. The gramian-based control structure selection methods is not limited to the effect of a single specified frequency. This category of techniques suggests better input-output pairing and provides the opportunity of selecting more complex controller structures. In the last few years the gramian-based control structure selection methods have been improved and generalized significantly [8]-[13].

The techniques for control structure selection and in particular the methods from the category of gramian-based control structure selection, which have been reported so far, mostly developed for linear systems. However, it is well-known that many industrial processes and plants are nonlinear and linear models cannot approximate such systems accurate enough. To address this issues nonlinear control structure selection methods are needed to be developed. Bilinear systems are among important classes of nonlinear systems which are known to describe a lot of industrial processes and plants accurately [14]-[16].

This paper focuses on control structure selection for discrete-time MIMO bilinear processes. The generalized cross-gramian is introduced in this paper for discrete-time bilinear processes and plants. The existence of the generalized cross-gramian is studied and it is shown that if the cross-gramian exists, it can be obtained by solving a generalized Sylvester equation. To solve the generalized Sylvester equation in a computationally efficient way, an iterative method is developed and presented. The generalized cross-gramians are computed for all SISO subsystems of the discrete-time MIMO bilinear systems. These gramians are used to build the generalized Hankel Interaction Index Array (HIIA) which is used for control structure selection. The generalized HIIA is a generalized version of the ordinary HIIA which was introduced for linear systems before. The proposed method for control structure selection is among the few methods supporting bilinear processes and plants and enjoys the advantages of gramian based methods.

In the proposed control structure selection method, for
The entries of the generalized HIIA one generalized Sylvester equation is solved, therefore the proposed control configuration selection method is computationally more efficient than its gramian-based counterparts in [11] and [12]. This is because in the techniques which were proposed in [11] and [12] two generalized Lyapunov equations have to be solved for each PM entries.

The notation used in this paper is as follows: \( M^* \) denotes transpose of matrix if \( M \in \mathbb{R}^{n \times m} \) and complex conjugate transpose if \( M \in \mathbb{C}^{n \times m} \). The \( \otimes \) stands for the Kronecker Product. \( \text{Struc}(\Pi) = [\pi_{ij}]_{p \times p} \) shows the structure of the system \( \Pi \) which is a symbolic array where \( \pi_{ij} = * \), if there exist a subsystem in \( \Pi \) with input \( u_j \) and output \( y_i \). Otherwise: \( \pi_{ij} = 0 \).

II. CONTROL STRUCTURE SELECTION FOR DISCRETE-TIME LINEAR PROCESSES

The concept of gramians plays an important role in gramian-based control structure selection. Gramians were first introduced in [17], [18] and later in [19]. The controllability gramian is used to check if the system is controllable or not, furthermore it is also used to determine the degree of controllability. Analogously, the observability gramian can be used to study the observability. The cross-gramian contains information on level of both controllability and observability at the same time [20] - [24].

Let a stable discrete-time SISO system be described by:

\[
\begin{cases}
  x(k+1) = Fx(k) + bu(k), & x(k) \in \mathbb{R}^n \\
  y(k) = cx(k) + du(k),
\end{cases}
\]

the cross-gramian is defined as:

\[
W_{co} := \sum_{i=0}^{+\infty} F^i bc F^i
\]

the cross-gramian \( W_{co} \) is the solution of the Sylvester equation:

\[
FW_{co}F - F + bc = 0.
\]

The HIIA is built from the cross-gramian of the elementary subsystems. Let \( \mathcal{L} \) be a discrete-time linear process or plant which is described by:

\[
\mathcal{L} : \begin{cases}
  x(k+1) = Ax(k) + Bu(k), \\
  y(k) = Cx(k),
\end{cases}
\]

where \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^m \).

For this discrete-time MIMO linear system, we have:

\[
B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix},
\]

\[
C = \begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_m
\end{bmatrix},
\]

A set of elementary SISO systems can be associated to the MIMO system, such that each SISO system has a single input \( u_j(k) \) and single output \( y_i(k) \). The state-space model of each elementary system is given by:

\[
\begin{cases}
  x(k+1) = Ax(k) + b_j u_j(k), \\
  y_i(k) = c_i x(k),
\end{cases}
\]

Let \( W_{ij}^c \) be the generalized cross-gramian for elementary subsystem (6). The HIIA \( \Sigma_H \) is defined as:

\[
\Sigma_H = [\psi_{ij}] \in \mathbb{R}^{m \times m}
\]

where

\[
\psi_{ij} = \sigma(W_{ij}^c).
\]

The HIIA can be used to determine the elementary subsystems which are more important in the description of MIMO processes, and in this way it helps choosing the suitable pairing and a suitable controller structure. For control structure selection, first the nominal system structure \( \mathcal{L}_n \) has to be obtained. The nominal model is obtained by setting some of the elementary subsystems to zero in the structure of the actual MIMO process.

The entries of the HIIA helps us to figure out which elementary subsystems we need to set to zero in the nominal model. When \( \psi_{ij} \) is small, the associated elementary subsystem to the pair \( (i, j) \) can be set as zero in the nominal model. When \( \psi_{ij} \) is large, \( \mathcal{L}_{ij} \) needs to be present in the nominal system. The suitability of the controller structure selection depends on how large the sum of the chosen \( \psi_{ij} \) elements is. The larger the sum of the chosen \( \psi_{ij} \) elements are, the more suitable the controller structure would be. It should be noted that if the structure of nominal system is \( \text{Struc}(\mathcal{L}_n) \), a simple controller structure to selection will always be: \( \text{Struc}(C) = \text{Struc}(\mathcal{L}_n^{-1}) \).

III. THE GENERALIZED CROSS-GRAMIAN FOR DISCRETE-TIME BILINEAR SYSTEMS

Let \( \Pi \) be a discrete-time bilinear dynamical systems which is described by:

\[
\Pi : \begin{cases}
  x(k+1) = Az(k) + N x(k) u(k) + bu(k), \\
  y(k) = cx(k),
\end{cases}
\]

where \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}, y(k) \in \mathbb{R} \).

We define the generalized cross-gramian as:

\[
W_{co} := \sum_{i=1}^{+\infty} \sum_{k_1=0}^{+\infty} \cdots \sum_{k_m=0}^{+\infty} P_i Q_i
\]

where:

\[
P_i(k_1) = A^{k_1} b, \quad Q_i(k_1) = c A^{k_1},
\]

\[
P_i(0,\ldots,0) = A^{k_1} N P_{i-1},
\]

\[
Q_i(0,\ldots,0) = Q_{i-1} N A^{k_1}.
\]

The generalized cross-gramian is the solutions to the generalized Sylvester equation. In order to show this, the
following lemma is needed.

**Lemma 1:** Let $A$ and $M$ be square matrices and $A$ be stable. If $X$ satisfies:
\[
X = \sum_{i=0}^{+\infty} A^i M A^i,
\]
(14)
then $X$ is the solution to:
\[
AXA - X + M = 0
\]
(15)

**Proof:**
Since $X = \sum_{i=0}^{+\infty} A^i M A^i$ and $A$ is stable, we have:
\[
AXA - X = A \sum_{i=0}^{+\infty} A^i M A^i - \sum_{i=0}^{+\infty} A^i M A^i
\]
(16)
\[
= \sum_{i=0}^{+\infty} A^{i+1} M A^{i+1} - \sum_{i=0}^{+\infty} A^i M A^i
\]
(17)
Since:
\[
\sum_{i=0}^{+\infty} A^{i+1} M A^{i+1} = \sum_{i=1}^{+\infty} A^i M A^i,
\]
\[
AXA - X = \sum_{i=1}^{+\infty} A^i M A^i - \sum_{i=0}^{+\infty} A^i M A^i = -M
\]
(18)

In the sequel, it is shown that cross-gramian can be obtained from the generalized Sylvester equation.

**Theorem 1:**
If $A$ is stable, $W_{co}$ which is defined in (10) is the solution to the following generalized Sylvester equation:
\[
AW_{co} A - W_{co} + NW_{co} N + bc = 0
\]
(19)

**Proof:**
Let:
\[
\tilde{W}_1 = \sum_{k_1=0}^{+\infty} P_1(k_1)Q_1(k_1),
\]
(20)
\[
\tilde{W}_i = \sum_{k_i}^{+\infty} \cdots \sum_{k_i}^{+\infty} P_i(k_1, k_2, \ldots, k_i)Q_i(k_1, k_2, \ldots, k_i)
\]
(21)
Therefore, we have:
\[
W_{co} = \sum_{i=1}^{\infty} \tilde{W}_i
\]
(22)
Form (11) and (20), we have:
\[
\tilde{W}_1 = \sum_{k_1=0}^{+\infty} A^{k_1} bc A^{k_1},
\]
(23)
with $X = \tilde{W}_1$ and $M = bc$, Lemma 1 applies and therefore:
\[
AW_{1} - \tilde{W}_1 + bc = 0.
\]
(24)
We follow the same procedure for $i > 1$. Form (12), (13) and (21), we have:
\[
\tilde{W}_i = \sum_{k_i=0}^{+\infty} \cdots \sum_{k_i=0}^{+\infty} A^{k_i} N P_{i-1} Q_{i-1} NA^{k_i}
\]
(25)
\[
= \sum_{k_i=0}^{+\infty} A^{k_i} N \tilde{W}_{i-1} NA^{k_i}
\]
(26)
with $X = \tilde{W}_i$ and $M = \tilde{W}_{i-1}$, Lemma 1 applies and we have:
\[
AW_{i} A - \tilde{W}_i + N \tilde{W}_{i-1} N = 0, \quad i \geq 2.
\]
(27)
If we add up all equations described in (24) and (27) we have:
\[
A \sum_{i=1}^{\infty} \tilde{W}_i A - \sum_{i=1}^{\infty} \tilde{W}_i + bc + \sum_{i=2}^{\infty} N \tilde{W}_{i-1} N = 0,
\]
(28)
equivalently we have:
\[
A \sum_{i=1}^{\infty} \tilde{W}_i A - \sum_{i=1}^{\infty} \tilde{W}_i + bc + N \sum_{i=1}^{\infty} \tilde{W}_{i} N = 0.
\]
(29)
Since $W_{co} = \sum_{i=1}^{\infty} \tilde{W}_i$, we have:
\[
AW_{co} A - W_{co} + bc + NW_{co} N = 0.
\]
(30)
The generalized cross-gramian is the solution to the generalized Sylvester equation, in the following the conditions for solvability of the generalized Sylvester equation is presented.

**Proposition 1.** The generalized Sylvester equation (19) is solvable and have a unique solution if and only if:
\[
W = A^* \otimes A - I \otimes I + N^* \otimes N
\]
(30)
is non-singular.

**Proof:**
Let $vec(.)$ be an operator which takes a matrix and converts it to a vector by stacking its columns on top of one another. This operator has the following interesting property [25]:
\[
vec(M_1 M_2 M_3) = (M_3^* \otimes M_1) vec(M_2)
\]
(31)
Taking $vec(.)$ from (19), we have:
\[
vec(AW_{co} A - W_{co} + NW_{co} N) = -vec(bc),
\]
(32)
Using the property (31), we get:
\[
(A^* \otimes A - I \otimes I + N^* \otimes N) vec(W_{co}) = -vec(bc).
\]
(33)
The generalized Sylvester equation (19) is solvable and has a unique solution if this linear equation is solvable and has a unique solution. On the other hand, this linear equation has unique solution if and only if:

$$W = A^* \otimes A - I \otimes I + N^* \otimes N$$

is non-singular.

It is important to note that the solution to the generalized Sylvester equation is not necessarily will be the generalized cross-gramian. In the sequel, we present conditions which guarantee the existence of the generalized cross-gramian as the solution of the generalized Sylvester equation.

We know that the zero-input $\Pi$ is asymptotically stable or $A$ is stable if and only if there exist $\rho \in [0,1)$ and $\alpha > 0$ such that [27]:

$$\|A^i\| \leq \alpha \rho^i, \quad i = 0, 1, 2,...$$

(34)

**Theorem 2.** The generalized cross-gramian exists if the following two conditions hold:

1) $A$ is stable
2) $\|N\| < \sqrt{1-\rho^2}$.

Proof:

From (11)-(13), we have:

$$P_i Q_i = A^i N P_{i-1} Q_{i-1} N A^{k_i}, \quad i \geq 2,$$

(35)

Using (34), it can easily be shown that:

$$\|P_i Q_i\| \leq \alpha^{2i} \rho^{2(k_i+k_{i-1}+...+k_1)} \|N\|^{2(i-1)} \|bc\|$$

(36)

Using (10), we have:

$$\|W_{co}\| \leq \sum_{i=1}^{+\infty} \sum_{k_i=0}^{+\infty} \sum_{k_{i-1}=0}^{+\infty} \sum_{k_1=0}^{+\infty} \|P_i Q_i\|$$

$$\leq \sum_{i=1}^{+\infty} \sum_{k_i=0}^{+\infty} \sum_{k_{i-1}=0}^{+\infty} \sum_{k_1=0}^{+\infty} \alpha^{2i} \rho^{2(k_i+k_{i-1}+...+k_1)} \|N\|^{2(i-1)} \|bc\|$$

$$= \frac{\|bc\| \|N\| \sum_{i=1}^{+\infty} \alpha^{2i}}{\|N\|^2} \left( \frac{1}{1-\rho^2} \right)^i \|N\|^{2(i-1)} \|bc\|$$

$$= \frac{\|bc\| \sum_{i=1}^{+\infty} \alpha^{2i} \|N\|^{2i}}{\|N\|^2} \left( \frac{1}{1-\rho^2} \right)^i$$

(37)

Therefore, if $\alpha^2 \|N\|^2 \left( \frac{1}{1-\rho^2} \right) < 1$ or equivalently $\|N\| < \sqrt{1-\rho^2/\alpha}$ the generalized cross-gramian exists.

To cope with the problem of the existence of the generalized cross-gramian, a scaling-based method can be used. See [26] and [8] for more information.

In the sequel, an iterative scheme for solving the generalized Sylvester equation is presented. The proposed iterative technique is computationally more efficient than solving the the generalized Sylvester equation directly.

If the generalized cross-gramian exists it can be obtained by:

$$W_{co} = \lim_{i \to \infty} X_i$$

(38)

where:

$$AX_1 A - X_1 + bc = 0,$$

$$AX_1 A - X_1 + NX_{i-1} N + bc = 0, \quad i = 2, 3,...$$

(39)

In this method an ordinary Sylvester equation has to be solved in each iteration.

IV. THE GENERALIZED HANKEL INTERACTION INDEX ARRAY FOR CONTROL STRUCTURE SELECTION OF DISCRETE-TIME BILINEAR SYSTEMS

In this section, first the HIIA is extended for bilinear systems. The generalized HIIA is defined using the generalized cross-gramian which has been introduced in the previous section. Let $\Pi_m$ be a discrete-time bilinear dynamical system which is described by:

$$\Pi_m : \begin{cases} x(k+1) = Ax(k) + \sum_{j=1}^{m} N_j x(k) u_j(k) + Bu(k), \\ y(k) = C x(k). \end{cases}$$

(40)

where $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^m$.

For this MIMO bilinear system, we can write:

$$B = [b_1 \ b_2 \ \cdots \ b_m],$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}.$$  

(41)

A set of elementary SISO systems can be associated to the MIMO system, such that each SISO system has a single input $u_j(k)$ and single output $y_j(k)$.

The state-space representation of each elementary system is given by:

$$\begin{cases} x(k+1) = Ax(k) + N_j x(k) u_j(k) + b_j u_j(k), \\ y_j(k) = c_j x(k). \end{cases}$$

(42)

Let $W_{ji}^{G}$ be the generalized cross-gramian for elementary subsystem (42). The generalized HIIA $\Sigma_{GH}$ is defined as:

$$\Sigma_{GH} = [\psi_{ij}] \in \mathbb{R}^{m \times m},$$

(43)

where

$$\psi_{ij} = \sigma(W_{ji}^{G}).$$

(44)

The generalized HIIA $\Sigma_{GH}$ can be used for control configuration selection in the same way as the ordinary HIIA. The method is illustrated further with the help of a numerical example in the next section.
V. An Illustrative Example

Consider a discrete-time bilinear model with the following standard state-space representation:

\[
x(k+1) = Ax(k) + \sum_{j=1}^{3} N_j x(k) u_j(k) + Bu(k),
\]

\[
y(k) = Cx(k).
\]

where:

\[
A = \begin{pmatrix}
-0.625 & 0.182 & 0.311 & 0.0594 & 0.147 \\
0.0285 & -0.578 & -0.244 & 0.0857 & -0.277 \\
-0.175 & 0.297 & -0.736 & 0.168 & 0.193 \\
0.18 & -0.109 & -0.0253 & -0.419 & -0.122 \\
-0.302 & 0.108 & -0.164 & 0.144 & -0.716
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
1.19 & 0.862 & 0 \\
-1.61 & 0.00116 & -2.32 \\
0 & -0.0708 & 0.0799 \\
-1.95 & -2.49 & -0.948 \\
1.02 & 0 & 0.411
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0.677 & 0 & 0.536 & -0.147 & 0 \\
0 & 0.101 & 0.898 & 1.01 & -1.27 \\
-0.691 & 0.826 & -0.132 & -2.12 & -0.383
\end{pmatrix}
\]

\[
N_1 = N_3 = \begin{pmatrix}
-0.1 & 0 & 0 & 0 & 0 \\
0 & -0.1 & 0 & 0 \\
0 & 0 & -0.1 & 0 \\
0 & 0 & 0 & -0.1
\end{pmatrix}
\]

\[
N_2 = \begin{pmatrix}
-0.2 & 0 & 0 & 0 & 0 \\
0 & -0.2 & 0 & 0 \\
0 & 0 & -0.2 & 0 \\
0 & 0 & 0 & -0.2
\end{pmatrix}
\]

We solve the generalized Sylvester equations for the subsystems, and form the generalized Hankel interaction index:

\[
\Sigma_{GH} = \begin{pmatrix}
6.749 & 9.205 & 9.902 \\
5.367 & 7.682 & 8.534 \\
5.166 & 6.95 & 8.004
\end{pmatrix}
\]

The structure of nominal model which this suggests for a decentralized control is:

\[
\text{Struc}(\Pi_n) = \begin{pmatrix}
0 & 0 & * \\
0 & * & 0 \\
* & 0 & 0
\end{pmatrix}
\]

This structure is associated to: \( \sum = \psi_{13} + \psi_{22} + \psi_{31} = 22.7507 \) and a simple controller which is suggested for this is:

\[
\text{Struc}(C) = \begin{pmatrix}
0 & 0 & * \\
0 & * & 0 \\
* & 0 & 0
\end{pmatrix}
\]

A more complicated model structure is:

\[
\text{Struc}(\Pi_\alpha) = \begin{pmatrix}
0 & 0 & * \\
0 & * & 0 \\
* & 0 & 0
\end{pmatrix}
\]

which is associated to: \( \sum = \psi_{13} + \psi_{22} + \psi_{31} + \psi_{12} = 31.9552 \) and a simple controller which is suggested for this is:

\[
\text{Struc}(C) = \begin{pmatrix}
0 & 0 & * \\
0 & * & 0 \\
* & 0 & 0
\end{pmatrix}
\]

VI. Conclusions

A new method for selecting control structure for discrete-time MIMO bilinear processes and plants is proposed in this paper. First, the cross-gramian is introduced for discrete-time bilinear systems. The existence of the generalized cross-gramian is studied. The generalized Sylvester equation is derived for computation of the generalized cross-gramian. An iterative algorithm to solve the generalized Sylvester equation is proposed which helps computing the generalized cross-gramian more efficiently. The proposed generalized cross-gramian is used to form generalized Hankel Interaction Index Array. The generalized Hankel Interaction Index Array is the extension of the ordinary Hankel Interaction Index Array for bilinear systems. The proposed interaction measure is used for control structure selection of discrete-time MIMO bilinear processes and plants. The proposed method for control structure selection is among the few methods supporting bilinear processes and plants, enjoys the advantages of gramian based methods and it is more efficient in terms of computations compared to its counterparts.

References


